Compactness results and applications to some “zero mass” elliptic problems

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1 Introduction and statement of the main results

In this paper we study the elliptic problem,

\[-\Delta v = f'(v) \quad \text{in } \Omega,\]

in the so called “zero mass case” that is, roughly speaking, when \(f''(0) = 0\).

A particular example is

\[-\Delta v = v^{\frac{N+2}{N-2}} \quad \text{in } \mathbb{R}^N,\]

with \(N \geq 3\). This problem has been studied very intensely (see \([4, 17, 25]\)) and we know the explicit expression of the positive solutions

\[
v(x) = \frac{[N(N - 2)\lambda^2]^{(N-2)/4}}{[\lambda^2 + |x - x_0|^2]^{(N-2)/2}}, \quad \text{with } \lambda \geq 0, \ x_0 \in \mathbb{R}^N.\]

If \(f\) is not the critical power, we are led to require particular growth conditions on the nonlinearity \(f\). In fact, while in the “positive mass case” (namely when \(f''(0) < 0\)) the natural functional setting is \(H^1(\Omega)\) and we have suitable compact embeddings just assuming a subcritical behavior of \(f\), in the “zero mass case” the problem is studied in \(\mathcal{D}^{1,2}(\Omega)\) that is defined as the completion of \(C_0^\infty(\Omega)\) with respect to the norm

\[\|u\| = \left(\int_\Omega |\nabla u|^2 \, dx\right)^{\frac{1}{2}}.\]

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In order to recover analogous compactness results, we need to assume that \( f \) is supercritical near the origin and subcritical at infinity.

With these assumptions on \( f \), the problem (1) has been dealt with by Berestycki & Lions [14–16], when \( \Omega = \mathbb{R}^N, N \geq 3 \), and existence and multiplicity results have been proved.

Recently, Benci & Fortunato [9] have introduced a new functional setting, namely the Orlicz space \( L^p + L^q \), which arises very simply from the growth conditions on \( f \) and seems to be the natural framework for studying “zero mass” problems as shown also by Pisani in [24].

Using this new functional setting, Benci & Micheletti in [10] studied the problem (1), with Dirichlet boundary conditions, in the case of exterior domain, namely when \( \mathbb{R}^N \setminus \Omega \) is contained into a ball \( B_\varepsilon \). Under suitable assumptions, if the ball radius \( \varepsilon \) is sufficiently small, they are able to prove the existence of a positive solution.

The functional setting introduced in [9] seems to be the natural one also for studying the nonlinear Schrödinger equations with vanishing potentials, namely

\[
-\Delta v + V(x)v = f'(v), \quad \text{in } \mathbb{R}^N, \tag{2}
\]

with

\[
\lim_{x \to \infty} V(x) = 0.
\]

Some existence results for such a problem have been found by Benci, Grisanti & Micheletti [11, 12] and by Ghimenti & Micheletti [19]. Equation (2) has been studied also in [7], where the potential \( V(x) = V(|x|) \) is required to be positive and in \( L^1(a, b) \), with \( 0 < a < b \).

Even if in a different context, we need also to mention the paper of Ambrosetti, Felli & Malchiodi [2], where problem (2) is studied when the nonlinearity \( f(v) \) is replaced by a function \( f(x, v) \) of the type \( K(x)v^p \), with \( K \) vanishing at infinity.

In this paper, we study problem (1) in two different situations. In Section 4, we look for complex valued solutions of the following problem

\[
-\Delta v = f'(v) \quad \text{in } \mathbb{R}^3, \tag{3}
\]

assuming that \( f \in C^1(\mathbb{C}, \mathbb{R}) \) satisfies the following assumptions:

(f1) \( f(0) = 0; \)

(f2) \( \exists M > 0 \) such that \( f(M) > 0; \)

(f3) \( \forall \xi \in \mathbb{C} : |f'(\xi)| \leq c \min(|\xi|^{p-1}, |\xi|^{q-1}); \)
Compactness results and “zero mass” elliptic problems

\[ (f4) \quad f(e^{i\alpha} \rho) = f(\rho), \text{ for all } \xi = e^{i\alpha} \rho \in \mathbb{C}; \]

where \( 1 < p < 6 < q \) and \( c > 0 \).

Observe that an example of function satisfying the previous hypotheses can be obtained as follows. Let us consider the function \( \tilde{f} : \mathbb{R}^+ \to \mathbb{R} \) defined as

\[ \tilde{f}(t) := \begin{cases} \alpha t^p + b & \text{if } t \geq 1 \\ t^q & \text{if } t \leq 1 \end{cases}, \]

with \( a, b \in \mathbb{R} \) chosen in order to have \( \tilde{f} \in C^1 \) and let us define \( f : \mathbb{C} \to \mathbb{R} \) as \( f(\xi) = \tilde{f}(|\xi|) \).

Introducing the cylindrical coordinates \((r, z, \theta)\), for all \( n \in \mathbb{Z} \), we look for solutions of the type

\[ v^n(x, y, z) = u^n(r, z) e^{in\theta} \quad \text{with } u^n \in \mathbb{R}. \] (4)

We obtain the following existence result for problem (3):

**Theorem 1.1.** Let \( f \) satisfy the hypotheses \((f1'-f4')\). Then there exists a sequence \((v^n)\) of complex-valued solutions of problem (3), such that, for every \( n \in \mathbb{Z} \),

\[ v^n(x, y, z) = u^n(r, z) e^{in\theta}, \text{ with } u^n \in \mathbb{R}. \]

Actually, an existence result in the same spirit of ours is present in [23]. However, in [23] the problem is studied using different tools and the details are omitted. Moreover in [13] an interesting physical interpretation has been given to the complex valued solutions of the equation (3) in the positive mass case. In fact there has been shown the strict relation between such solutions and the standing waves of the Schrödinger equation with nonvanishing angular momentum.

In Section 5, we study

\[ \begin{cases} -\Delta v = f'(v) \quad \text{in } \mathbb{R}^2 \times I, \\ v = 0 \quad \text{in } \mathbb{R}^2 \times \partial I, \end{cases} \] (5)

where \( I \) is a bounded interval of \( \mathbb{R} \) and \( f \in C^1(\mathbb{R}, \mathbb{R}) \) satisfies the following assumptions:

\[ (f1') \quad f(0) = 0; \]

\[ (f2') \quad \forall \xi \in \mathbb{R} : f(\xi) \geq c_1 \min(|\xi|^p, |\xi|^q); \]

\[ (f3') \quad \forall \xi \in \mathbb{R} : |f'(\xi)| \leq c_2 \min(|\xi|^{p-1}, |\xi|^{q-1}); \]

\[ (f4') \quad \text{there exists } \alpha \geq 2 \text{ such that } \forall \xi \in \mathbb{R} : \alpha f(\xi) \leq f'(\xi)\xi; \]
we will prove the following multiplicity result:

**Theorem 1.2.** Let $f$ satisfy the hypotheses (f1'-f4'). Then there exist infinitely many solutions with cylindrical symmetry of problem (5).

In order to approach to our problems, we use a functional framework related to the Orlicz space $L^p + L^q$. The main difficulty in dealing with such spaces consists in the lack of suitable compactness results. In view of this, the key points of this paper are two compactness theorems presented in Section 3. They are obtained adapting a well known lemma of Esteban & Lions [18] to our situation.

The paper is organized as follows: Section 2 is devoted to a brief recall on the space $L^p + L^q$; in Section 3, we present our compactness results; in Sections 4 and 5 we solve problems (3) and (5); finally, by using similar arguments as in Section 3, in the Appendix we are able to prove a compact embedding theorem which improves [7, Theorem 3.5].

2 Some properties of the $L^p + L^q$ spaces

In this section, we present some basic facts on the Orlicz space $L^p + L^q$. For more details, see [9, 20, 24].

Let $\Omega \subset \mathbb{R}^3$. For $1 < p < 6 < q$, denote by $(L^p(\Omega), \| \cdot \|_{L^p})$ and by $(L^q(\Omega), \| \cdot \|_{L^q})$ the usual Lebesgue spaces with their norms, and set

$$L^p + L^q(\Omega) := \{ v : \Omega \to \mathbb{R} | \exists (v_1, v_2) \in L^p(\Omega) \times L^q(\Omega) \text{ s.t. } v = v_1 + v_2 \}.$$ 

The space $L^p + L^q(\Omega)$ is a Banach space with the norm

$$\| v \|_{L^p + L^q(\Omega)} := \inf \{ \| v_1 \|_{L^p} + \| v_2 \|_{L^q} | (v_1, v_2) \in L^p(\Omega) \times L^q(\Omega), v_1 + v_2 = v \}$$

and its dual is the Banach space $(L^{p'}(\Omega) \cap L^{q'}(\Omega), \| \cdot \|_{L^{p'}(\Omega) \cap L^{q'}(\Omega)})$, where $p' = \frac{p}{p-1}$, $q' = \frac{q}{q-1}$ and

$$\| \varphi \|_{L^{p'}(\Omega) \cap L^{q'}(\Omega)} : \| \varphi \|_{L^{p'}} + \| \varphi \|_{L^{q'}}.$$

In the sequel, for all $v \in L^p + L^q(\Omega)$, we set

$$\Omega^> := \{ x \in \Omega \mid |v(x)| > 1 \}.$$ 

$$\Omega^\leq := \{ x \in \Omega \mid |v(x)| \leq 1 \}.$$ 

The following theorem summarizes some properties about $L^p + L^q$ spaces.
Compactness results and “zero mass” elliptic problems

Theorem 2.1. 1. Let \( v \in L^p + L^q(\Omega) \). Then

\[
\max \left( \|v\|_{L^q(\Omega^\leq)} - 1, \frac{1}{1 + \text{meas}(\Omega^>)^{1/r}} \|v\|_{L^p(\Omega^>)} \right) \\
\leq \|v\|_{L^p + L^q} \leq \max \left( \|v\|_{L^q(\Omega^\leq)}, \|v\|_{L^p(\Omega^>)} \right)
\]

(6)

where \( r = \frac{pq}{q - p} \).

2. The space \( L^p + L^q \) is continuously embedded in \( L^p_{\text{loc}} \).

3. For every \( r \in [p, q] : L^r(\Omega) \hookrightarrow L^p + L^q(\Omega) \) continuously.

4. The embedding

\[
\mathcal{D}^{1,2}(\Omega) \hookrightarrow L^p + L^q(\Omega)
\]

is continuous.

Proof

1. See Lemma 1 in [9].

2. See Proposition 6 of [24].

3. See Corollary 9 in [24].

4. It follows from the point 3 and the Sobolev continuous embedding

\[
\mathcal{D}^{1,2}(\Omega) \hookrightarrow L^6(\Omega).
\]

The following theorem has been proved in [24]:

Theorem 2.2. Let \( f \) be a \( C^1(\mathbb{C}, \mathbb{R}) \) function (resp. \( C^1(\mathbb{R}, \mathbb{R}) \)) satisfying assumption (f3) (resp. (f3')). Then the functional

\[
v \in L^p + L^q(\Omega) \mapsto \int_{\Omega} f(v) \, dx
\]

is of class \( C^1 \). Moreover the Nemytski operator

\[
f'(v) : v \in L^p + L^q(\Omega) \mapsto f'(v) \in (L^p + L^q(\Omega))'
\]

is bounded.

Using Theorem 2.2 we get a very useful inequality for the \( L^p + L^q \)-norm.
Theorem 2.3. For all $R > 0$, there exists a positive constant $c = c(R)$ such that, for all $v \in L^p + L^q(\Omega)$ with $\|v\|_{L^p + L^q} \leq R$,

$$\max \left( \int_{\Omega^>} |v|^p \, dx, \int_{\Omega^<} |v|^q \, dx \right) \leq c(R) \|v\|_{L^p + L^q}. \quad (8)$$

Proof. Let us introduce $g \in C^1(\mathbb{R}, \mathbb{R})$ such that $g(0) = 0$ and with the following growth conditions:

(g1) $\forall \xi \in \mathbb{R}: g(\xi) \geq c_1 \min(|\xi|^p, |\xi|^q)$;

(g2) $\forall \xi \in \mathbb{R}: |g'(\xi)| \leq c_2 \min(|\xi|^{p-1}, |\xi|^{q-1})$.

Integrating in (g1) we get

$$\int_{\Omega} g(v) \, dx \geq c_1 \left( \int_{\Omega^>} |v|^p \, dx + \int_{\Omega^<} |v|^q \, dx \right).$$

By Lagrange theorem, there exists $t \in [0, 1]$ such that

$$\int_{\Omega} g'(tv)v \, dx = \int_{\Omega} g(v) \, dx \geq c_1 \left( \int_{\Omega^>} |v|^p \, dx + \int_{\Omega^<} |v|^q \, dx \right).$$

Then, by the boundness of $g'$ (see Theorem 2.2), there exists $M > 0$ such that

$$M \|v\|_{L^p + L^q} \geq \int_{\Omega} |g'(tv)|v \, dx \geq c_1 \left( \int_{\Omega^>} |v|^p \, dx + \int_{\Omega^<} |v|^q \, dx \right)$$

and hence the conclusion. \qed

Remark 2.4. Combining the inequality (6) with the estimate (8) we deduce that the following statements are equivalent:

a) $v_n \to v$ in $L^p + L^q(\Omega)$,

b) $\|v_n - v\|_{L^p(\Omega^>_n)} \to 0$ and $\|v_n - v\|_{L^q(\Omega^<_n)} \to 0$,

where $\Omega^>_n = \{ x \in \Omega \mid |v_n(x) - v(x)| \geq 1 \}$ and $\Omega^<_n$ is analogously defined.
3 Compactness results

In this section we present the main tools of this paper, namely a compactness theorem for sequences with “a particular symmetry” and a compact embedding of a suitable subspace of $D^{1,2}$ into $L^p + L^q$. The proofs of these results are both modelled on that of Theorem 1 of [18], which states that a suitable subspace of $H^1$ is compactly embedded into $L^p$, for $p$ subcritical.

First of all, for every interval $I$ of $\mathbb{R}$, possibly unbounded, we introduce the following subspace of $D^{1,2}_cyl(\mathbb{R}^2 \times I)$:

$$D^{1,2}_cyl(\mathbb{R}^2 \times I) = \{ u \in D^{1,2}(\mathbb{R}^2 \times I) \mid u(\cdot, \cdot, z) \text{ is radial, for a.e. } z \in I \}.$$  

Moreover we assume the following

**Definition 3.1.** If $u : \mathbb{R}^2 \times I \to \mathbb{R}$ is a measurable function, we call $z$-symmetrical rearrangement of $u$ in $(x, y)$ the Schwarz symmetrical rearrangement of the function $u(x, y, \cdot) : z \in I \mapsto u(x, y, z) \in \mathbb{R}$.

Moreover we call $z$-symmetrical rearrangement of $u$ the function $v$ defined as follows

$$\tilde{u} : (x, y, z) \in \mathbb{R}^2 \times I \mapsto \tilde{u}_{x,y}(z)$$

where $\tilde{u}_{x,y}$ is the $z$-symmetrical rearrangement of $u$ in $(x, y)$.

In our first compactness result, we consider $I = \mathbb{R}$.

**Theorem 3.2.** Let $(u_j)_j$ be a bounded sequence in $D^{1,2}_cyl(\mathbb{R}^3)$ such that $u_j$ is the $z$-symmetrical rearrangement of itself. Then $(u_j)_j$ possesses a converging subsequence in $L^p + L^q(\mathbb{R}^3)$, for all $1 < p < 6 < q$.

**Proof** With an abuse of notations, in the sequel for every $v \in D^{1,2}_cyl(\mathbb{R}^3)$, we denote by $v$ also the function defined in $\mathbb{R}^+ \times \mathbb{R}$ as

$$v(\sqrt{x^2 + y^2}, z) = v(x, y, z).$$

Being the proof quite long and involved, we divide it into several steps, for reader’s convenience. Since $(u_j)_j$ is bounded in the $D^{1,2}(\mathbb{R}^3)$ norm, there exists $u \in D^{1,2}_cyl(\mathbb{R}^3)$ such that

$$u_j \rightharpoonup u \text{ weakly in } D^{1,2}_cyl(\mathbb{R}^3) \text{ and in } L^p + L^q(\mathbb{R}^3), \ 1 < p \leq 6 \leq q, \ (9)$$

$$u_j \to u \text{ a.e. in } \mathbb{R}^3, \ \ (10)$$

$$u_j \to u \text{ in } L^p(K), \text{ for all } K \subset \subset \mathbb{R}^3, \ 1 \leq p < 6. \ \ (11)$$
By Lions [21],

\[ \forall j \geq 1, \forall r > 0, z \neq 0 : \quad |u_j(x, y, z)| \leq \frac{C}{r^\frac{4}{q} |z|^\frac{4}{q}}, \tag{12} \]

where \( r = \sqrt{x^2 + y^2} \). By (12), for \( R > 0 \) large enough, \( j \geq 1 \) and for all \( (r, |z|) \in (R, +\infty) \times (R, +\infty) \), we have

\[
\begin{align*}
|u_j(x, y, z)| &< 1, \\
|u(x, y, z)| &< 1, \\
|(u_j - u)(x, y, z)| &< 1.
\end{align*}
\tag{13}
\]

Let

\[
\begin{align*}
D_1 &:= \{(r, z) \in \mathbb{R}^+ \times \mathbb{R} : r > R, |z| > R\}, \\
D_2 &:= \{(r, z) \in \mathbb{R}^+ \times \mathbb{R} : 0 \leq r \leq R, |z| \leq R\}, \\
D_3 &:= \{(r, z) \in \mathbb{R}^+ \times \mathbb{R} : 0 \leq r \leq R, |z| > R\}, \\
D_4 &:= \{(r, z) \in \mathbb{R}^+ \times \mathbb{R} : r > R, |z| \leq R\}.
\end{align*}
\]

Obviously \( \bigcup_{i=1}^{4} D_i = \mathbb{R}^+ \times \mathbb{R} \). Moreover denote by \( \chi_{D_i} \) the characteristic function of \( D_i \) and observe that, since

\[
\|u_j - u\|_{L^p + L^q} = \left\| \sum_{i=1}^{4} (u_j - u) \chi_{D_i} \right\|_{L^p + L^q} \leq \sum_{i=1}^{4} \|u_j - u\|_{L^p + L^q(D_i)} = \sum_{i=1}^{4} \|u_j - u\|_{L^p + L^q(D_i)},
\]

then we get the conclusion if we prove that, for all \( i = 1, \ldots, 4 \),

\[ u_j \to u \text{ in } L^p + L^q(D_i). \]

**Claim 1:** \( u_j \to u \text{ in } L^p + L^q(D_1) \).

Suppose for a moment that \( q > 8 \). By (13), for every \( (x, y, z) \in D_1 \), we have \( |(u_j - u)(x, y, z)| < 1 \), then the inequality (6) implies

\[ \|u_j - u\|_{L^p + L^q(D_1)} \leq \|u_j - u\|_{L^q(D_1)}. \tag{14} \]

On the other hand, since

\[ u_j \to u \text{ a.e. and } |(u_j - u)(r, z)|^q \leq \frac{C}{|r|^{\frac{q}{4}} |z|^{\frac{q}{4}}} \in L^1(D_1), \]

by Lebesgue theorem $u_j \to u$ in $L^q(D_1)$.
If $6 < q \leq 8$, then take $r \in (q - 6, 4(q - 6))$ and set $\alpha = \frac{6}{6-q+r}$ and $\beta = \frac{6}{q-r}$.
Observe that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ so, by Holder,
\begin{align*}
\int_{D_1} |u_j - u|^q \, dx \, dy \, dz & \leq \left( \int_{D_1} |u_j - u|^{\alpha r} \right)^{\frac{1}{\alpha}} \left( \int_{D_1} |u_j - u|^{(\beta q-r)} \right)^{\frac{1}{\beta}} \\
& \leq \left( \int_{D_1} |u_j - u|^{\frac{6r}{6-q+r}} \right)^{\frac{q}{6-q+r}} \|u_j - u\|_{L^6}^{q-r}.
\end{align*}
(15)
Since $(u_j)_j$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^3)$, it is bounded in $L^6(\mathbb{R}^3)$.
Moreover, since $q - 6 < r < 4(q - 6)$, certainly $\frac{6r}{6-q+r} > 8$, and then the last integral in inequality (15) goes to zero.
Hence the Claim 1 is proved.

**Claim 2:** $u_j \to u$ in $L^p + L^q(D_2)$.
It is enough to observe that, since $D_2$ has finite measure, $L^p + L^q(D_2) = L^p(D_2)$ (see [24, Remark 5]) and then we get the conclusion by (11).

**Claim 3:** $u_j \to u$ in $L^p + L^q(D_3)$.
First suppose $p < 4$ and consider $g \in C^1(\mathbb{R}, \mathbb{R})$, $g(0) = 0$, such that the following growth and strong convexity conditions hold
\begin{align*}
(\text{G}) & \exists k_1 > 0 \text{ s.t. } \forall t \in \mathbb{R} : |g'(t)| \leq k_1 \min(|t|^{p-1}, |t|^{q-1}), \\
(\text{SC}) & \exists k_2 > 0 \text{ s.t. } \forall s, t \in \mathbb{R} : g(s) - g(t) - g'(t)(s - t) \\
& \geq k_2 \min(|s - t|^p, |s - t|^q).
\end{align*}
Since $g(0) = 0$, from (G) and (SC) we deduce that
\begin{equation}
\exists k_3, k_4 > 0 \text{ s.t. } \forall s \in \mathbb{R} : k_3 \min(|s|^p, |s|^q) \leq g(s) \leq k_4 \min(|s|^p, |s|^q). \tag{16}
\end{equation}
The condition (SC) has been introduced in [5], where an explicit example of function satisfying (SC) is also given.
For almost every $(x, y) \in \mathbb{R}^2$, we set $u^{x,y} : \mathbb{R} \to \mathbb{R}$ defined as $u^{x,y}(z) := u(x, y, z)$. We give an analogous definition for $u^{x,y}_j$, for all $j \geq 1$.
For almost every $(x, y) \in \mathbb{R}^2$ with $(x^2 + y^2)^{1/2} \leq R$, we set
\begin{equation}
w_j(x, y) := \int_{(-R,R)^3} g(u^{x,y}_j(z)) \, dz.
\end{equation}
We show that
\[ w_j(x, y) \to \int_{(-R,R)^c} g(u^{x,y}(z)) \, dz \quad \text{for a.e. } (x, y) \in B_R. \tag{17} \]

Consider
\[ \left| w_j(x, y) - \int_{(-R,R)^c} g(u^{x,y}(z)) \, dz \right| \leq \int_{(-R,R)^c} |g(u^{x,y}_j) - g(u^{x,y})| \, dz \]
\[ = \int_{(-R,R)^c} |g'(\theta_j^{x,y})||u^{x,y}_j - u^{x,y}| \, dz \tag{18} \]

where, for almost every \((x, y) \in B_R\), \(\theta_j^{x,y}\) is a suitable convex combination of \(u^{x,y}_j\) and \(u^{x,y}\). Since \((u_j)_j\) is bounded in \(L^p + L^q\), \(g'(\theta_j^{x,y})\) is bounded in \((L^p + L^q)'\) (see Theorem 2.2) so, by (18), to prove (17) we are reduced to show that
\[ u^{x,y}_j \to u^{x,y} \quad \text{in } L^p + L^q((-R,R)^c) \quad \text{for a.e. } (x, y) \in B_R. \]

For, define
\[ \Omega_j^{x,y} = \{ z \in \mathbb{R} \mid |z| > R, \ |u^{x,y}_j(z) - u^{x,y}(z)| > 1 \} \]
so that, by (6),
\[ \|u_j^{x,y} - u^{x,y}\|_{L^p + L^q((-R,R)^c)} \leq \max \left( \|u_j^{x,y} - u^{x,y}\|_{L^p(\Omega_j^{x,y})}, \|u_j^{x,y} - u^{x,y}\|_{L^q((-R,R)^c \setminus \Omega_j^{x,y})} \right). \tag{19} \]

By Lebesgue theorem and by (12),
\[ \|u_j^{x,y} - u^{x,y}\|_{L^q((-R,R)^c \setminus \Omega_j^{x,y})} \to 0 \quad \text{for a.e. } (x, y) \in B_R. \tag{20} \]

Moreover there exists \(R' = R'(x, y) \in \mathbb{R}\) such that for all \(|z| > R'\)
\[ |u_j^{x,y}(z) - u^{x,y}(z)| \leq \frac{2C}{\sqrt{1/4}|z|^{1/4}} \leq 1. \]

Let \(\tilde{R} = \max(R, R')\). We have
\[ \|u_j^{x,y} - u^{x,y}\|_{L^p(\Omega_j^{x,y})} \leq \int_{(-R,R) \cup (\tilde{R}, R)} |u_j^{x,y} - u^{x,y}|^p \, dz \]
and then
\[ \|u_j^{x,y} - u^{x,y}\|_{L^p(\Omega_j^{x,y})} \to 0 \quad \text{for a.e. } (x, y) \in B_R, \tag{21} \]
Compactness results and “zero mass” elliptic problems

because \( u_j^{x,y} \to u^{x,y} \) in \( L^p(\{ z \in \mathbb{R} \mid R \leq |z| \leq \tilde{R} \}) \) by (11).

By (19), (20) and (21) we get
\[
\| u_j^{x,y} - u^{x,y} \|_{L^p + L^q((-R,R)^c)} \to 0 \quad \text{for a.e. } (x, y) \in B_R
\]
and, hence, (17).

We claim that the sequence \((w_j)_j\) is bounded in \( W^{1,1}(B_R) \). Indeed the \( L^1\)-norm of \( w_j \) is bounded since \((u_j)_j\) is bounded in \( D^{1,2} \) and then in \( L^6 \). Moreover, if we set
\[
\Omega_{u_j} := \{ (x, y, z) \in \mathbb{R}^3 \mid |u_j(x, y, z)| > 1 \},
\]
we have
\[
\| \nabla_{(x,y)} w_j \|_{L^1(B_R)} = \int_{B_R} \left| \nabla_{(x,y)} \left( \int_{(-R,R)^c} g(u_j(x, y, z)) \, dz \right) \right| \, dx \, dy
\]
\[
\leq \int_{B_R} \left( \int_{(-R,R)^c} |g'(u_j)||\nabla_{(x,y)} u_j| \, dz \right) \, dx \, dy
\]
\[
\leq \int_{D_3} |g'(u_j)||\nabla u_j| \, dx \, dy \, dz
\]
\[
\leq c_5 \left[ \int_{D_3 \setminus \Omega_{u_j}} |u_j|^{p-1} |\nabla u_j| \, dx \, dy \, dz + \int_{D_3 \setminus \Omega_{u_j}} |u_j|^{q-1} |\nabla u_j| \, dx \, dy \, dz \right]^{1/2}
\]
\[
= c_5 \left[ \left( \int_{D_3 \setminus \Omega_{u_j}} |u_j|^{2(p-1)} \, dx \, dy \, dz \right)^{1/2} \cdot \left( \int_{D_3 \setminus \Omega_{u_j}} |\nabla u_j|^2 \, dx \, dy \, dz \right)^{1/2}
\]
\[
+ \left( \int_{D_3 \setminus \Omega_{u_j}} |u_j|^{2(q-1)} \, dx \, dy \, dz \right)^{1/2} \cdot \left( \int_{D_3 \setminus \Omega_{u_j}} |\nabla u_j|^2 \, dx \, dy \, dz \right)^{1/2} \right]
\]
\[
\leq c_5 \left( \| u_j \|_{L^6}^2 \|\nabla u_j\|_{L^2} + \| u_j \|_{L^6}^2 \|\nabla u_j\|_{L^2} \right)
\]
\[
\leq c_5 \|\nabla u_j\|_{L^2}^4
\]
where we have used the fact that \( 2(p - 1) < 6 < 2(q - 1) \) and \( D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3) \). Since \( W^{1,1}(B_R) \) is compactly embedded into \( L^1(B_R) \), there exists \( w \in L^1(B_R) \) such that
\[
w_j \to w \quad \text{in } L^1(B_R),
\]
and, by (17),
$$w(x, y) = \int_{(-R,R)^c} g(u(x, y, z)) \, dz \quad \text{a.e. in } B_R. \quad (23)$$

By the definition of $w_j$ and (23),
$$\left| \int_{D_3} \left( g(u_j(x, y, z)) - g(u(x, y, z)) \right) \, dx \, dy \, dz \right| \leq \int_{B_R} \left| \int_{(-R,R)^c} \left( g(u_j(x, y, z)) - g(u(x, y, z)) \right) \, dz \right| \, dx \, dy$$
$$= \|w_j - w\|_{L^1(B_R)},$$
and so from (22) we deduce that
$$\int_{D_3} g(u_j(x, y, z)) \, dx \, dy \, dz \to \int_{D_3} g(u(x, y, z)) \, dx \, dy \, dz. \quad (24)$$

Now observe that, by (SC) we have
$$\int_{D_3} g(u_j) - \int_{D_3} g(u) - \int_{D_3} g'(u)(u_j - u) \geq \int_{D_{3,j}^\alpha} |u_j - u|^p + \int_{D_{3,j}^\beta} |u_j - u|^q,$$
where $D_{3,j}^\alpha = \{(x, y, z) \in D_3 \mid |u_j - u| > 1\}$ and $D_{3,j}^\beta$ is analogously defined. Moreover, by (G) and by Proposition 29 in [24],
$$g'(u) \in \left( L^p + L^q(D_3) \right)',$$
and then, from (9) and (24), we obtain
$$\int_{D_{3,j}^\alpha} |u_j - u|^p \to 0 \quad \text{for } 1 < p < 4, \quad (25)$$
$$\int_{D_{3,j}^\beta} |u_j - u|^q \to 0 \quad \text{for } q > 6. \quad (26)$$

If $4 \leq p < 6$, then consider $r \in (0, 2(6 - p))$, $\alpha = 6/(6 - p + r)$, and $\beta = 6/(p - r)$. Since $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, we have
$$\int_{D_{3,j}^\alpha} |u_j - u|^p \, dx \, dy \, dz$$
$$\leq \left( \int_{D_{3,j}^\alpha} |u_j - u|^{ar} \, dx \, dy \, dz \right)^{1/\alpha} \left( \int_{D_{3,j}^\beta} |u_j - u|^{(p-r)b} \, dx \, dy \, dz \right)^{1/\beta}$$
and, since $(p - r)\beta = 6$ and $\alpha r = \frac{6r}{6 - p + r} < 4$, from (25) we get

$$\int_{D_{3,j}} |u_j - u|^p \to 0 \quad \text{for all } 4 \leq p < 6.$$  \hfill (27)

From (25), (26), (27) and using inequality (6) we have

$$u_j \to u \text{ in } L^p + L^q(D_3), \quad \text{for all } 1 < p < 6 < q.$$

**Claim 4:** $u_j \to u$ in $L^p + L^q(D_4)$.

The arguments are analogous to those in the previous case.

The theorem is completely proved. \hfill \Box

By the previous theorem we can easily prove the following

**Theorem 3.3.** If $I \subset \mathbb{R}$ is bounded, then $\mathcal{D}^{1,2}_{cyl}(\mathbb{R}^2 \times I)$ is compactly embedded in $L^p + L^q(\mathbb{R}^2 \times I)$, for every $1 < p < 6 < q$.

**Proof** For the sake of simplicity, we denote $\Omega = \mathbb{R}^2 \times I$. Let $(v_j)_j \subset \mathcal{D}^{1,2}(\Omega)$ be a bounded sequence. Up to subsequences, there exists $v \in \mathcal{D}^{1,2}(\Omega)$ such that

$$v_j \to v \quad \text{weakly in } \mathcal{D}^{1,2}(\Omega) \text{ and in } L^p + L^q(\Omega), \ 1 < p \leq 6 \leq q,$$

$$v_j \to v \quad \text{a.e. in } \Omega,$$

$$v_j \to v \quad \text{in } L^p(K) \quad \text{for all } K \subset \subset \Omega, \ 1 \leq p < 6.$$  \hfill (28, 29, 30)

Let $(\hat{v}_j)_j$ be the sequence of the corresponding $z$-symmetrical rearrangements of $(v_j)_j$, then we get that there exists $w \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ such that, up to a subsequence,

$$\hat{v}_j \to w \quad \text{in } \mathcal{D}^{1,2}(\mathbb{R}^3),$$  \hfill (31)

and, by Theorem 3.2,

$$\hat{v}_j \to w \quad \text{in } L^p + L^q(\mathbb{R}^3).$$  \hfill (32)

We claim

1. $w$ is the $z$-symmetrical rearrangement of $v$;

2. $v_j \to v \text{ in } L^p + L^q(\Omega)$. 
1. For $R > 0$, we set $B_R$ the ball in $\mathbb{R}^2$ of radius $R$ and centered in the origin. Let $\hat{v}$ be the $z$-symmetrical rearrangement of $v$. Observe that $\hat{v}, \hat{v}_j \in L^p(B_R \times \mathbb{R})$, indeed $v, v_j \in L^p(B_R \times I)$, for all $j \geq 1$, and

$$\int_{B_R \times \mathbb{R}} |\hat{v}|^p = \int_{B_R \times I} |v|^p$$

$$\int_{B_R \times \mathbb{R}} |\hat{v}_j|^p = \int_{B_R \times I} |v_j|^p, \quad \text{for all } j \geq 1.$$

We deduce that $\hat{v}_{x,y} = \hat{v}(x, y, \cdot)$ and $\hat{v}_{x,y} = \hat{v}_j(x, y, \cdot)$ are in $L^p(\mathbb{R})$, for almost every $(x, y) \in B_R$ and for all $j \geq 1$.

Since the Schwarz symmetrization is a contraction in $L^p(\mathbb{R})$ (see, for example, [1]),

$$\|\hat{v}_j - \hat{v}\|^p_{L^p(B_R \times \mathbb{R})} = \int_{B_R} \|\hat{v}_j^{x,y} - \hat{v}^{x,y}\|^p_{L^p(\mathbb{R})} \, dx \, dy \leq \int_{B_R} \|v_j^{x,y} - v^{x,y}\|^p_{L^p(I)} \, dx \, dy = \|v_j - v\|^p_{L^p(B_R \times I)};$$

therefore, by (30),

$$(\hat{v}_j \to \hat{v}) \quad \text{in } L^p(B_R \times \mathbb{R}).$$

Since $R$ is arbitrary,

$$\hat{v}_j \to \hat{v} \quad \text{a.e. in } \mathbb{R}^3$$

so, by (32), $\hat{v} = w$.

2. Consider $g : \mathbb{R} \to \mathbb{R}$ as in the proof of Theorem 3.2. Since the functional

$$u \in L^p + L^q(\mathbb{R}^3) \longmapsto \int_{\mathbb{R}^3} g(u) \, dx$$

is continuous and we have proved that

$$\hat{v}_j \to \hat{v} \quad \text{in } L^p + L^q(\mathbb{R}^3),$$

we get

$$\int_{\Omega} g(v_j) = \int_{\mathbb{R}^3} g(\hat{v}_j) \to \int_{\mathbb{R}^3} g(\hat{v}) = \int_{\Omega} g(\hat{v}). \quad (33)$$

Using (SC), (28), (33) and (6), we deduce that

$$v_j \to v \quad \text{in } L^p + L^q(\Omega).$$

□
4 A complex-valued solutions problem

In this section we deal with the problem

$$-\Delta v = f'(v) \quad \text{in } \mathbb{R}^3,$$

(34)

assuming that $f \in C^1(\mathbb{C}, \mathbb{R})$ and satisfies the following assumptions:

(f1) $f(0) = 0$;

(f2) $\exists M > 0$ such that $f(M) > 0$;

(f3) $\forall \xi \in \mathbb{C} : |f'(\xi)| \leq c \min(|\xi|^{p-1}, |\xi|^{q-1})$;

(f4) $f(e^{i\alpha} \rho) = f(\rho)$, for all $\xi = e^{i\alpha} \rho \in \mathbb{C}$;

where $1 < p < 6 < q$ and $c > 0$.

For all $n \in \mathbb{Z}$, we look for solutions of the type $v^n(x, y, z) = u^n(r, z)e^{i\omega}$. Then, passing to cylindrical coordinates, since (f4) implies that $f'(e^{i\alpha} \rho) = f'(\rho)e^{i\alpha}$, by some computations one can check that, if $v^n(x, y, z)$ is solution of the problem (34), then $u^n(r, z)$ satisfies

$$-\frac{\partial}{\partial r} \left( r \frac{\partial u^n}{\partial r} \right) - r \frac{\partial^2 u^n}{\partial z^2} + \frac{n^2}{r} u^n = r f'(u^n) \quad \text{in } \mathbb{R}^+ \times \mathbb{R}.$$ (35)

Conversely, if $u^n(r, z)$ satisfies (35), then $v^n(x, y, z)$ is solution of (34) in $\mathbb{R}^3 \setminus \mathbb{R}_z$, where $\mathbb{R}_z$ is the z-axis.

In the sequel we will denote with $C_0^\infty(\mathbb{R}^+ \times \mathbb{R})$ the set of smooth functions with compact support.

Let us introduce the following Banach spaces:

- $L_r^s(\mathbb{R}^+ \times \mathbb{R})$ the completion of $C_0^\infty(\mathbb{R}^+ \times \mathbb{R})$ with respect to the norm

$$\|u\|_{L_r^s}^s := \int_{\mathbb{R}_r \times \mathbb{R}} r |u|^s dr dz;$$

- $(L^p + L^q)_r(\mathbb{R}^+ \times \mathbb{R}) := \{ u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{C} \mid \exists (u_1, u_2) \in L^p_r \times L^q_r \text{ s.t. } u = u_1 + u_2 \}$ with the norm

$$\|u\|_{(L^p + L^q)_r} := \inf \{ \|u_1\|_{L^p_r} + \|u_2\|_{L^q_r} \mid (u_1, u_2) \in L^p_r \times L^q_r, u = u_1 + u_2 \};$$

- $E_r(\mathbb{R}^+ \times \mathbb{R})$ the completion of $C_0^\infty(\mathbb{R}^+ \times \mathbb{R})$ with respect to the norm

$$\|u\|_r^2 := \int_{\mathbb{R}_r \times \mathbb{R}} \left[ r \left( \frac{\partial u}{\partial r} \right)^2 + r \left( \frac{\partial u}{\partial z} \right)^2 \right] dr dz.$$

\( E_{n,r}(\mathbb{R}^+ \times \mathbb{R}) \) the completion of \( C_0^\infty(\mathbb{R}^+ \times \mathbb{R}) \) with respect to the norm
\[
\|u\|_{n,r}^2 : \int_{\mathbb{R}^+ \times \mathbb{R}} \left[ r \left( \frac{\partial u}{\partial r} \right)^2 + r \left( \frac{\partial u}{\partial z} \right)^2 + \frac{n^2}{r} u^2 \right] \, dr \, dz.
\]

**Lemma 4.1.** The following embeddings are continuous:
\[
E_{n,r}(\mathbb{R}^+ \times \mathbb{R}) \hookrightarrow E_r(\mathbb{R}^+ \times \mathbb{R}) \hookrightarrow (L^p + L^q)_{r}(\mathbb{R}^+ \times \mathbb{R}).
\]

**Proof** The first one is trivial. The second one derives from the following argument: consider the spaces
\[
(L^p + L^q)_{cyl}(\mathbb{R}^3) := \{ u \in L^p + L^q(\mathbb{R}^3) \mid u(\cdot, \cdot, z) \text{ is radial, for a.e. } z \in \mathbb{R} \}
\]
and
\[
\mathcal{F} := \{ u : \mathbb{R}^3 \to \mathbb{R} \mid u(\cdot, \cdot, z) \text{ is radial, for a.e. } z \in \mathbb{R} \};
\]
if we denote by \( \mathbb{R}_z \) the \( z \)-axis and set \( \hat{u}(x, y, z) = u((x^2 + y^2)^{1/2}, z) \), by the map \( u(r, z) \to \hat{u}(x, y, z) \) we have
\[
E_r(\mathbb{R}^+ \times \mathbb{R}) \simeq C_0^\infty(\mathbb{R}^3 \setminus \mathbb{R}_z) \cap \mathcal{F} \hookrightarrow \mathcal{D}^{1,2}(\mathbb{R}^2 \times \mathbb{R}),
\]
\[
(L^p + L^q)_r(\mathbb{R}^+ \times \mathbb{R}) \simeq (L^p + L^q)_{cyl}(\mathbb{R}^2 \times \mathbb{R}),
\]
so the second embedding follows immediately from (36), (37) and (7). □

For all \( n \in \mathbb{Z} \), we will find a solution of equation (35) looking for critical points of the functional \( J_n : E_{n,r}(\mathbb{R}^+ \times \mathbb{R}) \to \mathbb{R} \) defined as
\[
J_n(u) := \frac{1}{2} \int_{\mathbb{R}^+ \times \mathbb{R}} \left[ r \left( \frac{\partial u}{\partial r} \right)^2 + r \left( \frac{\partial u}{\partial z} \right)^2 + \frac{n^2}{r} u^2 \right] \, dr \, dz,
\]
constrained on the manifold
\[
\Sigma_n := \left\{ u \in E_{n,r}(\mathbb{R}^+ \times \mathbb{R}) \mid \int_{\mathbb{R}^+ \times \mathbb{R}} r f(u) \, dr \, dz = 1 \right\}.
\]
By (f3) and by Theorem 2.2, the functional
\[
F_n : u \in E_{n,r}(\mathbb{R}^+ \times \mathbb{R}) \longmapsto \int_{\mathbb{R}^+ \times \mathbb{R}} r f(u) \, dr \, dz,
\]
is well defined and continuously differentiable.
Remark 4.2. Let us observe that $\Sigma$ is nonempty. Indeed, for $R > 2$, consider the functions

\[\alpha_R(t) := \begin{cases} \sqrt{M}(|t| - 1) & \text{if } 1 \leq |t| < 2, \\ \sqrt{M} & \text{if } 2 \leq |t| < R, \\ \sqrt{M}(R + 1 - |t|) & \text{if } R \leq |t| < R + 1, \\ 0 & \text{otherwise}, \end{cases}\]

and

\[\beta_R(t) := \begin{cases} \sqrt{M} & \text{if } 1 \leq |t| < R, \\ \sqrt{M}(R + 1 - |t|) & \text{if } R \leq |t| < R + 1, \\ 0 & \text{otherwise}, \end{cases}\]

and set $u_R(r, z) := \alpha_R(r)\beta_R(z)$. Of course, $u_R \in E_{n,r}(\mathbb{R}^+ \times \mathbb{R})$ and by similar arguments as in [14, Proof of Theorem 2] it can be shown that, for $R$ large enough,

\[\int_{\mathbb{R}^+ \times \mathbb{R}} rf(u_R) \, dr \, dz > 0.\]

Now, if $\sigma$ is a suitable rescaling parameter, the function

\[u_{R,\sigma} : (r, z) \mapsto u_R(\sigma r, \sigma z)\]

belongs to $\Sigma$. A crucial step for the proof of Theorem 1.1 is the following

Theorem 4.3. Let $f$ satisfy (f1-f4). Then, for all $n \in \mathbb{Z}$, there exists $u^n \in E_{n,r}(\mathbb{R}^+ \times \mathbb{R})$ such that

\[J_n(u^n) = \min_{\Sigma_n} J_n.\]

Proof. Let us fix $n \in \mathbb{Z}$. Let $(u^n_j)_j$ be a minimizing sequence for $J_n$ constrained on $\Sigma$, namely $(u^n_j)_j$ is contained in $\Sigma$ and satisfies

\[J_n(u^n_j) \to \inf_{\Sigma_n} J_n, \quad \text{as } j \to \infty.\]

Without lost of generality, we can suppose that, for all $j \geq 1$, $u^n_j$ coincides with its $z$-symmetrical rearrangement $\tilde{u}^n_j$. Otherwise, since also $(\tilde{u}^n_j)_j$ is contained in $\Sigma$ and, moreover,

\[J_n(\tilde{u}^n_j) \leq J_n(u^n_j),\]

...
we should simply replace \((u^n_j)_j\) by \((\tilde{u}^n_j)_j\). Since \((u^n_j)_j\) is a bounded sequence in \(E_{n,r}\), by Lemma 4.1 certainly there exists \(u^n \in E_{n,r}\) such that
\[
u^n_j \rightharpoonup u^n \text{ weakly in } E_{n,r} \text{ and in } (L^p + L^q)_r(\mathbb{R}_+ \times \mathbb{R}). \tag{38}\]

On the other hand, by Lemma 4.1 and (36), the sequence \(\hat{u}^n_j(x, y, z) = u^n_j((x^2 + y^2)^{\frac{1}{2}}, z)\) is bounded in \(D_{cyl}^{1,2}(\mathbb{R}^3)\), and then, from Theorem 3.2, we have that there exists \(\hat{u}^n \in (L^p + L^q)_{cyl}(\mathbb{R}^3)\) such that, up to a subsequence,
\[
\hat{u}^n_j \rightharpoonup \hat{u}^n \text{ in } (L^p + L^q)_{cyl}(\mathbb{R}^3). \tag{39}\]

By continuity,
\[
\int_{\mathbb{R}^3} f(\hat{u}^n_j) \to \int_{\mathbb{R}^3} f(\hat{u}^n), \quad \text{as } j \to \infty. \tag{40}\]

Moreover, comparing (38) and (39), by (37) we deduce that \(\hat{u}^n_j(x, y, z) = u^n((x^2 + y^2)^{\frac{1}{2}}, z)\), so, passing to cylindrical coordinates in (40), we have
\[
\int_{\mathbb{R}^+ \times \mathbb{R}} rf(u^n_j) \, dr \, dz \to \int_{\mathbb{R}^+ \times \mathbb{R}} rf(u^n) \, dr \, dz, \quad \text{as } j \to \infty,
\]
and then \(u^n \in \Sigma_n\).
Finally, the weak lower semicontinuity of the \(E_{n,r}\)-norm and (38) imply
\[
J_n(u^n) = \min_{\Sigma_n} J_n,
\]
and the theorem is proved. \(\square\)

Now we show how Theorem 1.1 follows immediately from Theorem 4.3.

**Proof of Theorem 1.1** Let us fix \(n \in \mathbb{Z}\). Let \(u^n \in E_{n,r}(\mathbb{R}_+ \times \mathbb{R})\) be a minimizer of \(J_n\) constrained on \(\Sigma_n\), whose existence is guaranteed by Theorem 4.3. Since \(u^n\) is a critical point of \(J_n\) constrained on \(\Sigma_n\), there exists a Lagrange multiplier \(\lambda^n\) such that \((\lambda^n, u^n)\) satisfies
\[
-\frac{\partial}{\partial r} \left( r \frac{\partial u^n}{\partial r} \right) - r \frac{\partial^2 u^n}{\partial z^2} + \frac{n^2}{r} u^n = \lambda^n \, r f'(u^n) \quad \text{in } \mathbb{R}^+ \times \mathbb{R},
\]
and so, as already observed
\[
-\Delta v^n = \lambda^n \, f'(v^n) \quad \text{in } \mathbb{R}^3 \setminus \mathbb{R}_z,
\]
for \( v^n(x, y, z) = u^n(r, z)e^{in\theta} \). Following the idea of [13] (see Theorem 3, therein), we argue that \((\lambda^n, v^n)\) is, in fact, a solution of

\[
-\Delta v^n = \lambda^n f'(v^n) \quad \text{in } \mathbb{R}^3.
\]

Let us observe that \( \lambda^n > 0 \). Indeed, by (f4), we can find \( w^n \in D^{1,2}(\mathbb{R}^3) \), of the type \( w^n(x, y, z) = \gamma^n(r, z)e^{in\theta} \), with \( \gamma^n(r, z) \in \mathbb{R} \), such that

\[
\int_{\mathbb{R}^3} f'(v^n) \bar{w}^n dx dy dz = \int_{\mathbb{R}^3} f'(u^n) \gamma^n dx dy dz > 0.
\]

Now we can repeat the arguments of [14, Theorem 2] to prove that \( \lambda^n > 0 \). Finally, it is easy to see that

\[
v_{\lambda^n}(x) := v^n \left( \frac{1}{\sqrt{\lambda^n}} x \right)
\]

is a solution of (34). \( \square \)

5 A “zero mass” problem in \( \mathbb{R}^2 \times I \)

In this section we consider the problem

\[
\begin{aligned}
-\Delta v &= f'(v) \quad \text{in } \mathbb{R}^2 \times I, \\
v &= 0 \quad \text{in } \mathbb{R}^2 \times \partial I,
\end{aligned}
\]

(41)

where \( I \) is a bounded interval of \( \mathbb{R} \) and \( f \in C^1(\mathbb{R}, \mathbb{R}) \) satisfies the following assumptions:

(f1’) \( f(0) = 0; \)

(f2’) \( \forall \xi \in \mathbb{R} : f'(\xi) \geq c_1 \min(\|\xi\|^p, \|\xi\|^q); \)

(f3’) \( \forall \xi \in \mathbb{R} : |f'(\xi)| \leq c_2 \min(\|\xi\|^{p-1}, \|\xi\|^{q-1}); \)

(f4’) there exists \( \alpha \geq 2 \) such that \( \forall \xi \in \mathbb{R} : \alpha f(\xi) \leq f'(\xi)\xi; \)

with \( 2 < p < 6 < q \) and \( c_1, c_2 > 0. \)

Set \( \Omega := \mathbb{R}^2 \times I. \) By (f3’) and Theorem 2.2, the functional

\[
J(v) := \frac{1}{2} \int_{\Omega} |\nabla v|^2 dx - \int_{\Omega} f(v) dx, \quad v \in D^{1,2}(\Omega),
\]
is $C^1$ and its critical points are weak solutions of (41).

In particular, we are interested in finding solutions with cylindrical symmetry. Since $D^{1,2}_{cyl}(\Omega)$ is a natural constraint, that is every critical point of the functional $J$ constrained on $D^{1,2}_{cyl}(\Omega)$ is a critical point of the nonconstrained functional, we will look for critical points of $J|_{D^{1,2}_{cyl}(\Omega)}$. To simplify the notations, from now on we set $\hat{J} = J|_{D^{1,2}_{cyl}(\Omega)}$.

Now we are ready to prove Theorem 1.2.

**Proof of Theorem 1.2** Since $\hat{J}$ is even and of class $C^1$, we may apply a very well known symmetrical version of the mountain pass theorem (see [3] or [8]). We just have to verify the following three conditions:

1. $J$ satisfies the Palais-Smale condition, i.e. any sequence $(v_j)_j$ in $D^{1,2}_{cyl}(\Omega)$ such that 
   \[ \left( \hat{J}(v_j) \right)_j \text{ is bounded, } \hat{J}'(v_j) \to 0, \]  
   admits a convergent subsequence;

2. there exist $\rho > 0$ and $C > 0$ such that 
   \[ \hat{J}(u) > C, \text{ for all } u \in S_\rho, \]
   where $S_\rho := \{ u \in D^{1,2}_{cyl}(\Omega) \mid \|u\| = \rho \}$;

3. for all $V \subset D^{1,2}_{cyl}(\Omega)$ such that $\dim V < +\infty$, we have 
   \[ \lim_{u \to +\infty \atop u \in V} \hat{J}(u) = -\infty. \]

For the proof of the 2nd and 3rd conditions, we refer to [24, Propositions 33 and 34].

As regards the Palais-Smale condition, we first observe that, by standard arguments, the hypotheses (42) imply that the sequence $(v_j)_j$ is bounded in $D^{1,2}_{cyl}(\Omega)$. So there exists $v \in D^{1,2}_{cyl}(\Omega)$ such that, up to a subsequence, 

\[ v_j \rightharpoonup v \text{ weakly in } D^{1,2}_{cyl}(\Omega), \]

and, by Theorem 3.3,

\[ v_j \to v \text{ in } L^p + L^q(\Omega). \]

(43)

Now, since 

\[ \hat{J}'(v_j) = -\Delta v_j - f'(v_j), \]

from the second of (42), we deduce that there exists an infinitesimal sequence $(\varepsilon_j)_j$ such that 

\[ -\Delta v_j = f'(v_j) + \varepsilon_j \text{ in } (D^{1,2}_{cyl}(\Omega))'. \]
Thus, inverting the Laplacian and using (43) and the continuity of the Nemytski operator associated with \( f' \), we have

\[
v_j = (-\Delta)^{-1}(f'(v_j) + \varepsilon_j) \to (-\Delta)^{-1}(f'(v)),
\]

and we are done. \( \square \)

## A Appendix

This section is entirely devoted to the proof of the following compact embedding theorem

**Theorem A.1.** Let \( N = \sum_{i=1}^{m} N_i \), with \( N \geq 3 \), \( m \geq 1 \) and \( N_i \geq 2 \) for all \( 1 \leq i \leq m \). Then the space

\[
D^{1,2}_s(\mathbb{R}^N) := \left\{ u \in D^{1,2}(\mathbb{R}^N) \mid \forall i \in \{1, \ldots, m\}, \forall x_i^0 \in \mathbb{R}^{N_i},
\begin{align*}
1 &\leq \frac{N}{2}, \\
|u(x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_m)| &\text{ is radial}
\end{align*}
\right\}
\]

is compactly embedded in \( L^p + L^q(\mathbb{R}^N), \) for all \( 1 < p < 2^* < q \), where \( 2^* = \frac{2N}{N-2} \).

First of all, let us observe that, when \( m = 1 \), Theorem A.1 has been already proved in [9]. In this paper, we will deal with the case \( m \geq 2 \). This theorem is used in [5] to find solutions for the semilinear Maxwell equations in even dimension.

**Remark A.2.** Let \( H^1_s(\mathbb{R}^N) \subset H^1(\mathbb{R}^N) \) be defined in an analogous way. In [22], Lions proved that \( H^1_s(\mathbb{R}^N) \) is compactly embedded into \( L^p(\mathbb{R}^N) \) for \( 2 < p < 2^* \).

To prove Theorem A.1 first we need to introduce two preliminary lemmas.

**Lemma A.3.** Let \( u \in D^{1,2}_s(\mathbb{R}^N) \) be such that \( u \) is decreasing with respect to \( |x_i| \), for \( i \geq 2 \). Then

\[
0 \leq u \leq C_N \left( \|u\|_{L^{2^*_N}}^{N/(2N-2)} \|\nabla x_1 u\|_{L^2}^{(N-2)/(2N-2)} \right) \cdot \cdot |x_1|^{-(N_1-1)(N-2)/(2N-2)} \prod_{i=2}^{m} |x_i|^{-N_i(N-2)/(2N-2)},
\]

where \( C_N > 0 \) depends only on \( N \).
The proof is simply a combination of [21, Proposition 2.1] and [22, Lemma 3.1].

**Lemma A.4.** Let $m \geq 1$, $n \geq 2$ and $N = m + n$. Let $(u_j)_j$ be a bounded sequence in $\mathcal{D}^{1,2}(\mathbb{R}^m \times \mathbb{R}^n)$, such that, for all $j \geq 1$ and for almost every $x \in \mathbb{R}^m$, the function $u_j(x, \cdot)$ is radial in $\mathbb{R}^n$. Then there exists $u \in \mathcal{D}^{1,2}(\mathbb{R}^m \times \mathbb{R}^n)$ such that for all $\Omega \subset \mathbb{R}^m$ bounded and with Lipschitz boundary $(u_j)_j$ converges, up to subsequences, to $u$ in $L^p + L^q(\Omega \times \mathbb{R}^n)$, for $1 < p < 2^* < q$.

**Proof** Since $(u_j)_j$ is a bounded sequence in $\mathcal{D}^{1,2}(\mathbb{R}^m \times \mathbb{R}^n)$, there exists $u \in \mathcal{D}^{1,2}(\mathbb{R}^m \times \mathbb{R}^n)$ such that

\[
\begin{align*}
&u_j \rightharpoonup u \text{ weakly in } \mathcal{D}^{1,2}(\mathbb{R}^m \times \mathbb{R}^n), \\
&u_j \to u \text{ a.e. in } \mathbb{R}^m \times \mathbb{R}^n. 
\end{align*}
\]  

(45)

Following an idea of [6], we set

\[
w_j = \begin{cases} |u_j - u| & \text{if } |u_j - u| \geq \varepsilon, \\
\varepsilon^{-2}|u_j - u|^3 & \text{if } |u_j - u| \leq \varepsilon,
\end{cases}
\]  

(46)

with $0 < \varepsilon < 1$. Observe that $w_j$ is well defined and almost everywhere differentiable. By some computations we get

\[
\begin{align*}
|w_j|^2 &\leq \varepsilon^{-4}|u_j - u|^6, \\
|\nabla w_j|^2 &\leq 9|\nabla u_j - \nabla u|^2,
\end{align*}
\]

so the sequence $(w_j)_j$ is bounded in $H^1(\mathbb{R}^m \times \mathbb{R}^n)$.

Now, fix $\Omega \subset \mathbb{R}^m$ bounded, with Lipschitz boundary and, with an abuse of notations, relabel by $w_j, u_j$ and $u$ the restrictions of the same functions to $\Omega \times \mathbb{R}^n$. By [22, Lemma 3.2], there exists $w \in H^1(\Omega \times \mathbb{R}^n)$ such that

\[
\begin{align*}
w_j &\to w \text{ in } L^r(\Omega \times \mathbb{R}^n), \text{ for } 2 < r < 2^*, \\
w_j &\to w \text{ a.e. in } \Omega \times \mathbb{R}^n.
\end{align*}
\]

On the other hand, by (45), we infer that $w = 0$. Let us consider for a moment $p > 2$. As in [6], we get

\[
\begin{align*}
\int_{\{|u_j - u| \geq 1\}} |u_j - u|^p \, dx + \int_{\{|u_j - u| \leq 1\}} |u_j - u|^q \, dx &\leq 2 \int_{\{|u_j - u| \geq \varepsilon\}} |u_j - u|^p \, dx + \int_{\{|u_j - u| \leq \varepsilon\}} |u_j - u|^q \, dx \\
&\leq 2 \int_{\{|u_j - u| \geq \varepsilon\}} |w_j|^p \, dx + \varepsilon^{q-2^*} \int_{\{|u_j - u| \leq \varepsilon\}} |u_j - u|^{2^*} \, dx \\
&\leq 2 \|w_j\|_{L^p(\Omega \times \mathbb{R}^m)}^{2^*} + \varepsilon^{q-2^*} \|u_j - u\|_{L^{2^*}(\Omega \times \mathbb{R}^m)}^{2^*}.
\end{align*}
\]
If, instead, $1 < p \leq 2$, taking $2 < r < 2^*$, we have:

$$
\int_{\{ |u_j-u| \geq 1 \}} |u_j-u|^p \, dx + \int_{\{ |u_j-u| \leq 1 \}} |u_j-u|^q \, dx
\leq \int_{\{ |u_j-u| \geq 1 \}} |u_j-u|^p \, dx + \int_{\{ |u_j-u| \leq 1 \}} |u_j-u|^q \, dx
\leq 2 \| w_j \|_{L^r(\Omega \times \mathbb{R}^m)}^{2^*} + \varepsilon^{q-2^*} \| u_j - u \|_{L^2(\Omega \times \mathbb{R}^m)}^{2^*}.
$$

Therefore, in any case, we can conclude that for all $1 < p < 2^*$, we have

$$
\int_{\{ |u_j-u| \geq 1 \}} |u_j-u|^p \, dx + \int_{\{ |u_j-u| \leq 1 \}} |u_j-u|^q \, dx
\leq 2 \| w_j \|_{L^r(\Omega \times \mathbb{R}^m)}^{2^*} + \varepsilon^{q-2^*} \| u_j - u \|_{L^2(\Omega \times \mathbb{R}^m)}^{2^*},
$$

with a suitable $2 < r < 2^*$. Since $w_j \to 0$ in $L^r(\Omega \times \mathbb{R}^m)$, for all $2 < r < 2^*$, and $(u_j)$ is bounded in $L^{2^*}(\Omega \times \mathbb{R}^m)$, by the arbitrariness of $\varepsilon$, we infer that

$$
\int_{\{ |u_j-u| \geq 1 \}} |u_j-u|^p \, dx + \int_{\{ |u_j-u| \leq 1 \}} |u_j-u|^q \, dx \to 0.
$$

Thus the conclusion follows from (6).

Now we pass to prove Theorem A.1.

**Proof of Theorem A.1** Let $(u_j)_j$ be a bounded sequence in $\mathcal{D}^{1,2}_s(\mathbb{R}^N)$. Up to a subsequence, we have

- $u_j \rightharpoonup u$ weakly in $\mathcal{D}^{1,2}_s(\mathbb{R}^N)$,
- $u_j \to u$ in $L^p(K)$, $\forall K \subset \subset \mathbb{R}^N$, $1 \leq p < 2^*$,
- $u_j \to u$ a.e. in $\mathbb{R}^N$.

Let $v_j = S_2(S_3(\ldots (S_m(u_j)) \ldots ))$, where $S_i$ is the symmetrizing operator with respect to $x_i \in \mathbb{R}^N$, with $i = 2, \ldots, m$.

**STEP 1:** There exists $v \in \mathcal{D}^{1,2}_s(\mathbb{R}^N)$ such that, up to a subsequence, $v_j \to v$ in $L^p + L^q(\mathbb{R}^N)$.

We will just sketch the proof, since it can modelled on that of Theorem 3.2.

First of all, since $(v_j)_j$ is bounded in $\mathcal{D}^{1,2}_s(\mathbb{R}^N)$, there exists $v \in \mathcal{D}^{1,2}_s(\mathbb{R}^N)$ such that

- $v_j \rightharpoonup v$ weakly in $\mathcal{D}^{1,2}_s(\mathbb{R}^N)$,
- $v_j \to v$ in $L^p(K)$, $\forall K \subset \subset \mathbb{R}^N$, $1 \leq p < 2^*$,
- $v_j \to v$ a.e. in $\mathbb{R}^N$. 

□
Let us observe that by (44), there exists $R > 0$ such that if $|x_i| > R$, for all $i = 1, \ldots, m$, then

$$
|v_j(x_1, \ldots, x_m) - v(x_1, \ldots, x_m)| < 1,
$$

$$
|v_j(x_1, \ldots, x_m)| < 1,
$$

$$
|v(x_1, \ldots, x_m)| < 1.
$$

Arguing as in the proof of Theorem 3.2, we need only to check that

$$
v_j \to v \quad \text{in } L^p + L^q(D_{I_1,I_2}),
$$

where

$$
D_{I_1,I_2} := \{x = (x_1, \ldots, x_m) \in \mathbb{R}^N \mid |x_i| \geq R, \text{ if } i \in I_1; \ |x_i| \leq R, \text{ if } i \in I_2\},
$$

for every $I_1$ and $I_2$ such that $I_1 \cap I_2 = \emptyset$ and $I_1 \cup I_2 = \{1, \ldots, m\}$.

Let us define, in particular, the following sets:

$$
D' := \{x \in \mathbb{R}^N \mid |x_i| \geq R, \text{ for all } i = 1, \ldots, m\},
$$

$$
D'' := \{x \in \mathbb{R}^N \mid |x_i| \leq R, \text{ for all } i = 1, \ldots, m\},
$$

$$
D''' := \{x \in \mathbb{R}^N \mid |x_i| \leq R, \text{ if } i = 1, \ldots, k, \ |x_i| \geq R, \text{ if } i = k + 1, \ldots, m\}.
$$

Without loss of generality, we need to prove the $L^p + L^q$-convergence of $(v_j)_j$ only in these three particular domains.

Arguing as in the Claim 1 of the proof of Theorem 3.2, we can prove that $v_j \to v$ in $L^p + L^q(D')$. Indeed, if $q > \frac{N_1(2N-2)}{(N_1-1)(N-2)}$, the convergence follows immediately by (44) and by Lebesgue theorem. If, instead, $2^* < q \leq \frac{N_1(2N-2)}{(N_1-1)(N-2)}$, then take

$$
r \in \left( q - 2^*, \frac{N_1(N-1)(q-2^*)}{N-N_1} \right),
$$

and set $\alpha = \frac{2^*}{2^*-q+r}$ and $\beta = \frac{2^*}{q-r}$. Observe that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ so, by Holder,

$$
\int_{D'} |v_j - v|^q \, dx \int_{D'} |v_j - v|^r |v_j - v|^{q-r} \, dx
\leq \left( \int_{D'} |v_j - v|^{\alpha r} \right)^{\frac{1}{\alpha}} \left( \int_{D'} |v_j - v|^{\beta r} \right)^{\frac{1}{\beta}}
\leq \left( \int_{D'} |v_j - v|^2 \right)^{\frac{2^*-q+r}{2^*}} \|v_j - v\|_{L^2}^{q-r}.
$$

(47)
Following the scheme of Claim 3 in the proof of Theorem 3.2, we show growth conditions \((G)\) and \((SC)\) satisfy the following conditions:

\((G)\) \(\exists c_1 > 0\) s.t. \(\forall t \in \mathbb{R} : |g'(t)| \leq c_1 \min(|t|^{p-1}, |t|^{q-1}),\)

\((SC)\) \(\exists c_2 > 0\) s.t. \(\forall s, t \in \mathbb{R} : g(s) - g(t) - g'(t)(s - t) \geq c_2 \min(|s - t|^p, |s - t|^q).\)

Moreover, for all \(\bar{x} = (x_1, \ldots, x_k)\), we set

\[ w_j(\bar{x}) = \int_{|x_{k+1}| \geq R} \cdots \int_{|x_m| \geq R} g(v_j(\bar{x}, x_{k+1}, \ldots, x_m)) \, dx_{k+1} \cdots dx_m. \]

Following the scheme of Claim 3 in the proof of Theorem 3.2, we show that \(v_j \rightharpoonup v\) in \(L^p + L^q(D''')\). Finally, if \(1 < p \leq \frac{2N-2}{N-2}\), we get the conclusion again by H"older inequality.

Therefore Step 1 is completely proved.

**STEP 2:** \(u_j \rightharpoonup u\) in \(L^p + L^q(\mathbb{R}^N).\)

Define \(\tilde{v}_j = S_3(S_4(\ldots (S_m(u_j)) \ldots)).\) Observe that \(v_j = S_2(\tilde{v}_j)\). Consider \(R > 0\) and

\[ Q_R = \{ x = (x_1, \ldots, x_m) \in \mathbb{R}^N | |x_1| \leq R, x_2 \in \mathbb{R}^{N_2}, |x_3| \leq R, \ldots, |x_m| \leq R \}. \]

By Lemma A.4, we have that there exists \(\tilde{v} \in \mathcal{D}_s^{1,2}(\mathbb{R}^N)\) such that, up to subsequences,

\[
\begin{align*}
\tilde{v}_j & \rightharpoonup \tilde{v} \quad \text{weakly in } \mathcal{D}_s^{1,2}(\mathbb{R}^N) \text{ and in } L^p + L^q(\mathbb{R}^N), \\
\tilde{v}_j & \to \tilde{v} \quad \text{ in } L^p + L^q(Q_R).
\end{align*}
\]

Let us show that \(S_2(\tilde{v}) = v\). Let \(g \in C^1(\mathbb{R}, \mathbb{R})\) be even and satisfying the growth conditions \((G)\) and \((SC)\). By [1, Corollary 2.3], we have

\[
\int_{Q_R} g(S_2(\tilde{v}_j) - S_2(\tilde{v})) = \int_{Q_R} g\left(|S_2(\tilde{v}_j) - S_2(\tilde{v})|\right)
\leq \int_{Q_R} g\left(|\tilde{v}_j| - |\tilde{v}|\right)
\leq \int_{Q_R} g(|\tilde{v}_j - \tilde{v}|) = \int_{Q_R} g(\tilde{v}_j - \tilde{v}).
\]
By (49), (G) and Theorem 2.2 we deduce that
\[ \int_{Q_R} g(\tilde{v}_j - \tilde{v}) \to 0. \]

Therefore
\[ \int_{Q_R} g\left(S_2(\tilde{v}_j) - S_2(\tilde{v})\right) \to 0 \]
and then, by (8),
\[ v_j = S_2(\tilde{v}_j) \to S_2(\tilde{v}) \text{ in } L^p + L^q(Q_R). \]

By Step 1 and by the arbitrariness of \( R > 0 \), we infer that \( S_2(\tilde{v}) = v \).
Hence
\[ \int_{\mathbb{R}^N} g(\tilde{v}_j) = \int_{\mathbb{R}^N} g(v_j) \to \int_{\mathbb{R}^N} g(v) = \int_{\mathbb{R}^N} g(\tilde{v}), \]
so, by (48) and (SC), we conclude that
\[ \tilde{v}_j \to \tilde{v} \text{ in } L^p + L^q(\mathbb{R}^N). \]

Iterating this argument we show that \( v = S_2(S_3(\ldots (S_m(u)) \ldots )) \) and hence the conclusion.

\[ \Box \]

**References**


